

# On degree-colorings of multigraphs

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## Abstract

A notion of degree-coloring is introduced; it captures some, but not all properties of standard edge-coloring. We conjecture that the smallest number of colors needed for degree-coloring of a multigraph  $G$  [the degree-coloring index  $\tau(G)$ ] equals  $\max\{\Delta, \omega\}$ , where  $\Delta$  and  $\omega$  are the maximum vertex degree in  $G$  and the multigraph density, respectively. We prove that the conjecture holds iff  $\tau(G)$  is a monotone function on the set of multigraphs.

## 1 Introduction.

The **chromatic index**  $\chi'(G)$  of a multigraph  $G(V, E)$  is the minimal number of colors (positive integers) that can be assigned to the edges of  $G$  so that no two adjacent edges receive the same color. Clearly,  $\Delta(G) \leq \chi'(G)$ , where  $\Delta(G)$  is the maximal vertex degree in  $G$ . The famous result by Vizing ([10]) establishes  $\chi' = \chi'(G) \leq \Delta(G) + p(G)$ , where  $p(G)$  is the maximal number of parallel edges in  $G$ . For graphs, in particular,  $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ . The problem of computing the exact value of the chromatic index was proved by Holyer ([5]) to be NP-hard even for cubic graphs. It is suspected that for every multigraph with  $\chi'(G) > \Delta(G) + 1$ , its chromatic index is determined by the parameter  $\omega(G)$ , called the multigraph **density**:

$$\omega(G) = \max_{H \subseteq G} \left\lceil \frac{e(H)}{\lfloor v(H)/2 \rfloor} \right\rceil,$$

where  $H$  is a sub-multigraph of  $G$ , and  $v(H)$  (resp.  $e(H)$ ) denotes the number of vertices (resp. edges) in  $H$ . It is easy to prove that  $\omega(G) \leq \chi'(G)$  for every multigraph  $G$ . Seymour in [7] and Stahl in [8] proved the equality  $\max(\Delta(G), \omega(G)) = \chi'^*(G)$ , where  $\chi'^*(G)$  is the fractional chromatic index of  $G$ , known to be polynomially computable (see [6]).

The following variation of the multigraph density idea was considered in [3]. Let  $\pi(F)$  denote the size of a maximum matching composed of the edges in a set  $F \subseteq E$ . Denote

$\omega^*(G) = \max_{F \subseteq E} \lceil \frac{|F|}{\pi(F)} \rceil$ . Then, it is easy to see that

$$\omega(G) \leq \omega^*(G) \leq \chi'(G).$$

It turns out (see [3]) that  $\omega^*(G) = \max(\Delta(G), \omega(G))$ .

Conjectures connecting  $\chi'(G)$ ,  $\omega(G)$ , and  $\Delta(G)$  were independently proposed by Goldberg ([1]) and Seymour ([7]) more than 30 years ago ([4],[9]). ([9]). Currently, the strongest variation of the conjecture ([2]) is as follows:

**Conjecture 1** *If  $\Delta(G) \neq \omega(G)$ , then  $\chi'(G) = \max(\Delta(G), \omega(G))$ , else  $\chi'(G) \leq \Delta(G) + 1$ .*

Every edge-coloring with colors  $1, 2, \dots, c$  yields an assignment  $\mu : V \rightarrow 2^{[1,c]}$ , where for every  $x \in V$ ,  $\mu(x)$  denotes the set of colors used on the edges incident to  $x$ . Given  $S \subseteq V$  and  $i \in [1, c]$ , the set of vertices  $x \in S$  such that  $i \in \mu(x)$  is denoted  $S^{(i)}(\mu)$ . It is easy to prove that the assignment  $\mu$  originated by an edge-coloring using colors  $1, \dots, c$  satisfies the following three conditions:

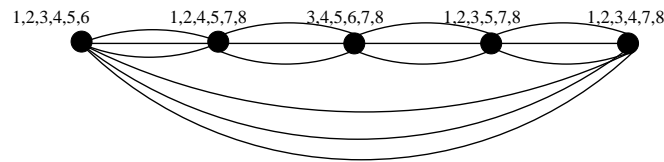
**Degree condition:**  $\forall x \in V(G), |\mu(x)| = \deg_G(x)$ ;

**Cover condition:**  $\forall S \subseteq V, |E(S)| \leq \sum_{i=1}^c \lfloor \frac{|S^{(i)}(\mu)|}{2} \rfloor$ ;

**Matching condition:**  $\forall i \in [1, c]$ , the submultigraph induced on  $V^{(i)}$  either has a perfect matching, or is empty.

**Definition 1** *An assignment  $\mu : V(G) \rightarrow 2^{[1,c]}$  satisfying the degree and the cover conditions is called a **degree-coloring**.*

Straightforward checking of the assignment presented in the Figure below shows that the assignment is a degree-coloring of the multicycle  $C$ . However, it is not originated by any edge-coloring of  $C$ , since the submultigraph of  $C$  induced on  $V^{(6)}$  has no perfect matching.



Let  $\tau(G)$  denote the smallest integer  $c$  for which a degree-coloring of  $G$  exists. It is easy to prove

**Lemma 1**  $\max(\Delta(G), \omega(G)) \leq \tau(G) \leq \chi'(G)$ .

**Conjecture 2** *(the  $\tau$ -conjecture): For every multigraph  $G$ ,  $\tau(G) = \max(\Delta(G), \omega(G))$ .*

A real-valued function  $\kappa(G)$  defined on the set of multigraphs is called **monotone** if for any multigraph  $G$  and any submultigraph  $H \subseteq G$ ,  $\kappa(H) \leq \kappa(G)$ . Clearly,  $\Delta(G)$  and  $\omega(G)$  are monotone functions.

**Conjecture 3** *The degree-coloring index  $\tau(G)$  is a monotone function on multigraphs.*

It is easy to see that Conjecture 2 implies Conjecture 3. We prove in this paper that the reverse is also true: the monotonicity of  $\tau(G)$  implies conjecture 2.

We use the standard graph-theoretical terminology which can be found in [11].

## 2 Monotonicity of $\tau(G)$ and the $\tau$ -conjecture.

It is easy to construct a  $\tau(G)$ -degree-coloring for a regular multigraph  $G$  with  $\omega(G) \leq \Delta(G)$ .

**Lemma 2** *If  $G$  is a  $\Delta$ -regular multigraph, and  $\omega(G) \leq \Delta$ , then  $\tau(G) = \Delta$ .*

**Proof.** From the definition,  $\tau(G) \geq \Delta$ . Consider the following assignment:

$$\forall x \in V(G), \mu(x) = \{1, 2, \dots, \Delta\}.$$

Given  $S \subseteq V(G)$ ,  $\forall i \in [1, \Delta]$ ,  $S^{(i)}(\mu) = S$ . Thus,

$$\sum_{i=1}^{\Delta} \lfloor \frac{|S^{(i)}(\mu)|}{2} \rfloor = \lfloor \frac{|S|}{2} \rfloor \Delta.$$

Since  $\omega(G) \leq \Delta$ , for any  $S \subseteq V$ ,  $\lfloor \frac{|S|}{2} \rfloor \Delta \geq \lfloor \frac{|S|}{2} \rfloor \omega(G) \geq |E(S)|$  implying  $\tau(G) = \Delta$ . ▮

Constructing a degree-coloring for a non-regular multigraph can be done via operation Regularization which, for every multigraph  $G$ , creates a regular multigraph  $R(G)$  containing  $G$  as an induced sub-multigraph.

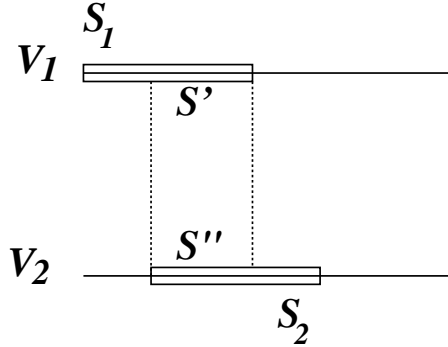
**Regularization:** If a multigraph  $G$  is regular and  $\omega(G) \leq \Delta(G)$ , then  $R(G) = G$ ; else

1. generate a disjoint isomorphic copy  $G' = (V', E')$  of  $G(V, E)$  with an isomorphic mapping  $f : V \rightarrow V'$  from  $G$  onto  $G'$ ;
2. let  $V(R(G)) = V \cup V'$  and initialize  $E(R(G))$  by setting  $E(R(G)) = E(G) \cup E(G')$ ;
3.  $\forall x \in V$ , add  $\max(\Delta(G), \omega(G)) - \deg_G(x)$  new edges  $xf(x)$  to  $E(R(G))$ .

**Lemma 3**  $\forall G, \omega(G) \leq \omega(R(G)) \leq \max(\omega(G), \Delta(G))$  and  $\Delta(R(G)) = \max(\Delta(G), \omega(G))$ .

**Proof.** If  $G = R(G)$ , the lemma is obvious. Let  $G \neq R(G)$ . Denote  $\Delta = \Delta(G)$ ,  $\omega = \omega(G)$ , and  $\rho = \max(\Delta, \omega)$ . Obviously,  $\Delta(R(G)) = \rho$  and  $\omega \leq \omega(R(G))$ .

To prove  $\omega(R(G)) \leq \rho$ , denote  $R = R(G)$ ,  $V(R) = V_1 \cup V_2$ , where  $V_1 = V(G)$  and  $V_2 = V(G')$ . Let  $f$  be an isomorphic mapping from  $V_1$  onto  $V_2$ . Given  $S \subseteq V(R)$ , let  $S_1 = S \cap V_1$ ,  $S_2 = S \cap V_2$ ,  $S' = S_1 \cap f^{-1}(S_2)$ , and  $S'' = S_2 \cap f(S_1)$ . Note that  $|S'| = |S''|$  and  $|E(S')| = |E(S'')|$ .



$$\begin{aligned}
\text{Then } |E(S)| &= |E(S_1)| + |E(S_2)| + \sum_{x \in S'} (\rho - \deg_G(x)) \\
&= |E(S_1 - S')| + |E(S_1 - S', S')| + |E(S')| + \\
&\quad |E(S_2 - S'')| + |E(S_2 - S'', S'')| + |E(S'')| + |S'| \rho - \sum_{x \in S'} \deg_G(x).
\end{aligned}$$

It is easy to check that

$$\begin{aligned}
&|E(S_1 - S', S')| + |E(S')| + |E(S_2 - S'', S'')| + |E(S'')| = \\
&|E(S_1 - S', S')| + |E(S_2 - S'', S'')| + 2|E(S')| \leq \sum_{x \in S'} \deg_G(x),
\end{aligned}$$

which yields the following upper bound

$$\begin{aligned}
|E(S)| &\leq |E(S_1 - S')| + |E(S_2 - S'')| + |S'| \rho \\
&\leq \lfloor \frac{|S_1| - |S'|}{2} \rfloor \rho + \lfloor \frac{|S_2| - |S''|}{2} \rfloor \rho + |S'| \rho.
\end{aligned}$$

To prove

$$\lfloor \frac{|S_1| - |S'|}{2} \rfloor \rho + \lfloor \frac{|S_2| - |S''|}{2} \rfloor \rho + |S'| \rho \leq \lfloor \frac{|S_1| + |S_2|}{2} \rfloor \rho,$$

note that it is straightforward if  $|S_1| + |S_2|$  is even. If  $|S_1| + |S_2|$  is odd, one out of two integers  $|S_1| - |S'|$  and  $|S_2| - |S''|$  is even and one is odd. Thus,

$$\lfloor \frac{|S_1| - |S'|}{2} \rfloor \rho + \lfloor \frac{|S_2| - |S''|}{2} \rfloor \rho + |S'| \rho = \frac{|S_1|}{2} \rho + \frac{|S_2|}{2} \rho - \frac{1}{2} \rho.$$

Since  $|S_1| + |S_2|$  is odd,

$$\lfloor \frac{|S_1| + |S_2|}{2} \rfloor \rho = \frac{|S_1| + |S_2|}{2} \rho - \frac{1}{2} \rho,$$

which implies the result.  $\blacksquare$

**Theorem 1** *If function  $\tau(G)$  is monotone on the set of all multigraphs, then for any multigraph  $G$ ,*

$$\tau(G) = \max\{\Delta(G), \omega(G)\}.$$

**Proof.** By Lemma 1,  $\max\{\Delta(G), \omega(G)\} \leq \tau(G)$ . On the other hand, since  $G \subseteq R(G)$ , it follows from Lemma 2 that  $\tau(G) \leq \tau(R(G)) = \max\{\Delta(G), \omega(G)\}$ .  $\blacksquare$

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